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Matrix Manipulations for Properties of Pell p-Numbers and their Generalizations

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Abstract

In this paper, we define the Pell-Pell p-sequence and then we discuss the connection of the Pell-Pell p-sequence with Pell and Pell p-sequences. Also, we provide a new Binet formula and a new combinatorial representation of the Pell-Pell p-numbers by the aid of the nth power of the generating matrix the Pell-Pell p-sequence. Furthermore, we obtain an exponential representation of the Pell-Pell p-numbers and we develop relationships between the Pell-Pell p-numbers and their permanent, determinant and sums of certain matrices.

1 Introduction

The well-known Pell sequence $\{P_n\}$ is defined by the following recurrence relation:

$$P_{n+2} = 2P_{n+1} + P_n \text{ for } n \geq 0 \text{ in which } P_0 = 0 \text{ and } P_1 = 1.$$

The generalized Pell (p,i)-numbers $\{P_p(n)\}$ for any given $p\,(p = 1, 2, 3, \ldots)$ is defined [14] by the following recurrence equation:

$$P_p^{(i)}(n) = 2P_p^{(i)}(n-1) + P_p^{(i)}(n-p-1)$$

for $n > p+1$ and $0 \leq i \leq p$, with initial conditions $P_p^{(i)}(1) = \cdots = P_p^{(i)}(i) = 0$ and $P_p^{(i)}(i+1) = \cdots = P_p^{(i)}(p+1) = 1$. When $i = 0$ and $p = 1$, the

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generalized Pell (p,i)-numbers $\{P_p(n)\}$ is reduced to the usual Pell sequence $\{P_n\}$.

It is easy to see that the characteristic polynomials of the Pell sequence and the Pell p-sequence are $f_1(x) = x^2 - 2x - 1$ and $f_2(x) = x^{p+1} - 2x^p - 1$, respectively. We use these in the next section.

Let the $(n+k)$th term of a sequence be defined recursively by a linear combination of the preceding k terms:

$$a_{n+k} = c_0 a_n + c_1 a_{n+1} + \cdots + c_{k-1} a_{n+k-1}$$

in which $c_0, c_1, \ldots, c_{k-1}$ are real constants. In [13], Kalman derived a number of closed-form formulas for the generalized sequence by the companion matrix method as follows:

Let the matrix A be defined by

$$A = [a_{i,j}]_{k\times k} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \ddots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ c_0 & c_1 & c_2 & & c_{k-2} & c_{k-1} \end{bmatrix},$$

then

$$A^n \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_{k-1} \end{bmatrix} = \begin{bmatrix} a_n \\ a_{n+1} \\ \vdots \\ a_{n+k-1} \end{bmatrix}$$

for $n \geqslant 0$.

Number theoretic properties such as those obtained from homogeneous linear recurrence relations relevant to this paper have been studied recently by many authors: see for example, [2, 5, 8, 9, 10, 11, 12, 21, 22, 23, 26]. In [1, 6, 7, 14, 15, 16, 17, 18, 24, 25, 27], the authors defined some linear recurrence sequences and gave their various properties by matrix methods. In this paper, we discuss connections between the Pell and Pell p-numbers. Firstly, we define the Pell-Pell p-sequence and then we give recurrence relation among this sequence, the Pell and Pell p-sequences. Also, we give the relations between the generating matrix of the Pell-Pell p-numbers and the elements of the Pell and Pell p-sequences. Furthermore, using the generating matrix of the Pell-Pell p-sequence, we obtain some new structural properties of the Pell p-numbers such as the Binet formula and the combinatorial representations. Finally, we obtain an exponential representation of the Pell-Pell p-numbers and we derive relationships between the Pell-Pell p-numbers and their sums, and permanents and, determinants of certain matrices.

2 The Main Results

Now we define the Pell-Pell p-sequence by the following homogeneous linear recurrence relation for any given $p\,(3,4,5,\ldots)$ and $n \geq 0$

$$P^{P,p}_{n+p+3} = 4P^{P,p}_{n+p+2} - 3P^{P,p}_{n+p+1} - 2P^{P,p}_{n+p} + P^{P,p}_{n+2} - 2P^{P,p}_{n+1} - P^{P,p}_{n} \quad (1)$$

in which $P^{P,p}_0 = \cdots = P^{P,p}_{p+1} = 0$ and $P^{P,p}_{p+2} = 1$.

First, we consider the relationship between the above the Pell-Pell p-sequence, Pell and Pell p-sequences.

Theorem 2.1. *Let P_n, $P_p\,(n)$ and $P^{P,p}_n$ be the nth Pell number, Pell p-number and Pell-Pell p-numbers, respectively, then*

$$P_n = P^{P,p}_{n+p-1} - P^{P,p}_n + P_p\,(n+p-1)$$

for $p \geq 3$ and $n \geq 0$.

Proof. The assertion may be proved by induction on n. It is clear that $P_0 = P^{P,p}_{p-1} - P^{P,p}_0 + P_p\,(p-1) = 0$. Suppose that the equation holds for $n > 0$. Then we must show that the equation holds for $n+1$. Since the characteristic polynomial of the Pell-Pell p-sequence $\left\{P^{P,p}_n\right\}$, is

$$q\,(x) = x^{p+3} - 4x^{p+2} + 3x^{p+1} + 2x^p - x^2 + 2x + 1$$

and

$$q\,(x) = f_1\,(x)\, f_2\,(x)$$

where $f_1\,(x)$ and $f_2\,(x)$ are the characteristic polynomials of the Pell sequence and the Pell p-sequence, respectively, we obtain the following relations:

$$P_{n+p+3} = 4P_{n+p+2} - 3P_{n+p+1} - 2P_{n+p} + P_{n+2} - 2P_{n+1} - P_n$$

and

$$P_p\,(n+p+3) = 4P_p\,(n+p+2) - 3P_p\,(n+p+1) - 2P_p\,(n+p) + P_p\,(n+2) - 2P_p\,(n+1) - P_p\,(n)$$

for $n > 0$. Thus, the conclusion is obtained. $\square$

From the recurrence relation (1), we have

$$\begin{bmatrix} P^{P,p}_{n+p+2} \\ P^{P,p}_{n+p+1} \\ P^{P,p}_{n+p} \\ \vdots \\ P^{P,p}_{n} \end{bmatrix} \begin{bmatrix} 4 & -3 & -2 & 0 & \cdots & 0 & 0 & 1 & -2 & -1 \\ 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 & 0 & 0 & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} P^{P,p}_{n+p+3} \\ P^{P,p}_{n+p+2} \\ P^{P,p}_{n+p+1} \\ \vdots \\ P^{P,p}_{n+1} \end{bmatrix}$$

for the Pell-Pell p-sequence $\left\{P^{P,p}_{n}\right\}$. Now we define

$$A_p = \begin{bmatrix} 4 & -3 & -2 & 0 & \cdots & 0 & 0 & 1 & -2 & -1 \\ 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 & 0 & 0 & 0 & 0 \\ \ddots & \ddots & \ddots & \ddots & \ddots & & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 1 & 0 \end{bmatrix}_{(p+3)\times(p+3)}.$$

The companion matrix $A_p = [a_{i,j}]_{(p+3)\times(p+3)}$ is said to be the Pell-Pell p-matrix. For more details on the companion type matrices, see [19, 20]. From induction on n, we get

$$(A_p)^n = \begin{bmatrix} P^{P,p}_{n+p+2} & -4P^{P,p}_{n+p+1} + P_n + P_{n-p+1} & -2P^{P,p}_{n+p+1} + P_p(n+1) & P_p(n+2) & \cdots & \\ P^{P,p}_{n+p+1} & -4P^{P,p}_{n+p} + P_{n-1} + P_{n-p} & -2P^{P,p}_{n+p} + P_p(n) & P_p(n+1) & \cdots & \\ P^{P,p}_{n+p} & -4P^{P,p}_{n+p-1} + P_{n-2} + P_{n-p-1} & -2P^{P,p}_{n+p-1} + P_p(n-1) & P_p(n) & \cdots & A^*_p \\ \vdots & \vdots & \vdots & \vdots & \cdots & \\ P^{P,p}_{n+1} & -4P^{P,p}_{n} + P_{n-p-1} + P_{n-2p} & -2P^{P,p}_{n} + P_p(n-p) & P_p(n-p+1) & \cdots & \\ P^{P,p}_{n} & -4P^{P,p}_{n-1} + P_{n-p-2} + P_{n-2p-1} & -2P^{P,p}_{n-1} + P_p(n-p-1) & P_p(n-p) & \cdots & \end{bmatrix},$$

for $n \geq 2p+1$, where

$$A_p^* = \begin{bmatrix} P_p(n+p-1) & -2P_{n+p+1}^{P,p} - P_{n+p}^{P,p} & -P_{n+p+1}^{P,p} \\ P_p(n+p-2) & -2P_{n+p}^{P,p} - P_{n+p-1}^{P,p} & -P_{n+p}^{P,p} \\ P_p(n+p-3) & -2P_{n+p-1}^{P,p} - P_{n+p-2}^{P,p} & -P_{n+p-1}^{P,p} \\ \vdots & \vdots & \vdots \\ P_p(n-2) & -2P_{n}^{P,p} - P_{n-1}^{P,p} & -P_{n}^{P,p} \\ P_p(n-3) & -2P_{n-1}^{P,p} - P_{n-2}^{P,p} & -P_{n-1}^{P,p} \end{bmatrix}.$$

It is clear that $detA_p = (-1)^{p+1}$.

In [14], Kılıc gave a Binet formula for the Pell p-numbers by matrix method. Now we concentrate on finding another Binet formula for the Pell-Pell p-numbers by the aid of the matrix $(A_p)^n$.

Lemma 2.1. *The characteristic equation of all the Pell-Pell p-numbers $x^{p+3} - 4x^{p+2} + 3x^{p+1} + 2x^p - x^2 + 2x + 1 = 0$ does not have multiple roots for $p \geq 3$.*

Proof. It is clear that $x^{p+3} - 4x^{p+2} + 3x^{p+1} + 2x^p - x^2 + 2x + 1 = \left(x^{p+1} - 2x^p - 1\right)\left(x^2 - 2x - 1\right)$. In [14], it was shown that the equation $x^{p+1} - 2x^p - 1 = 0$ does not have multiple roots for $p > 1$. It is easy to see that the roots of the equation $x^2 - 2x - 1 = 0$ are $1+\sqrt{2}$ and $1-\sqrt{2}$. Since $\left(1+\sqrt{2}\right)^{p+1} - 2\left(1+\sqrt{2}\right)^p - 1 \neq 0$ and $\left(1-\sqrt{2}\right)^{p+1} - 2\left(1-\sqrt{2}\right)^p - 1 \neq 0$ for $p > 1$, the equation $x^{p+3} - 4x^{p+2} + 3x^{p+1} + 2x^p - x^2 + 2x + 1 = 0$ does not have multiple roots for $p \geq 3$. $\square$

Let $x_1, x_2, \ldots, x_{p+3}$ be the roots of the equation $x^{p+3} - 4x^{p+2} + 3x^{p+1} + 2x^p - x^2 + 2x + 1 = 0$ and let V_p be a $(p+3) \times (p+3)$ Vandermonde matrix as follows:

$$V_p = \begin{bmatrix} (x_1)^{p+2} & (x_2)^{p+2} & \ldots & (x_{p+3})^{p+2} \\ (x_1)^{p+1} & (x_2)^{p+1} & \ldots & (x_{p+3})^{p+1} \\ \vdots & \vdots & & \vdots \\ x_1 & x_2 & \ldots & x_{p+3} \\ 1 & 1 & \ldots & 1 \end{bmatrix}.$$

Assume that $V_p(i,j)$ is a $(p+3) \times (p+3)$ matrix derived from the Vandermonde matrix V_p by replacing the j^{th} column of V_p by $W_p(i)$, where, $W_p(i)$ is a $(p+3) \times 1$ matrix as follows:

$$W_p = \begin{bmatrix} (x_1)^{n+p+3-i} \\ (x_2)^{n+p+3-i} \\ \vdots \\ (x_{p+3})^{n+p+3-i} \end{bmatrix}.$$

Theorem 2.2. *Let p be a positive integer such that $p \geq 3$ and let $(A_p)^n = [a_{i,j}^{(p,n)}]$ for $n \geq 1$, then*

$$a_{i,j}^{(p,n)} = \frac{\det V_p^{(i,j)}}{\det V_p}.$$

Proof. Since the equation $x^{p+3} - 4x^{p+2} + 3x^{p+1} + 2x^p - x^2 + 2x + 1 = 0$ does not have multiple roots for $p \geq 3$, the eigenvalues of the Pell-Pell p-matrix A_p are distinct. Then, it is clear that A_p is diagonalizable. Let $D_p = diag\,(x_1, x_2, \ldots, x_{p+3})$, then we may write $A_p V_p = V_p D_p$. Since the matrix V_p is invertible, we obtain the equation $(V_p)^{-1} A_p V_p = D_p$. Therefore, A_p is similar to D_p; hence, $(A_p)^n V_p = V_p (D_p)^n$ for $n \geq 1$. So we have the following linear system of equations:

$$\begin{cases} a_{i,1}^{(p,n)} (x_1)^{p+2} + a_{i,2}^{(p,n)} (x_1)^{p+1} + \cdots + a_{i,p+3}^{(p,n)} = (x_1)^{n+p+3-i} \\ a_{i,1}^{(p,n)} (x_2)^{p+2} + a_{i,2}^{(p,n)} (x_2)^{p+1} + \cdots + a_{i,p+3}^{(p,n)} = (x_2)^{n+p+3-i} \\ \vdots \\ a_{i,1}^{(p,n)} (x_{p+3})^{p+2} + a_{i,2}^{(p,n)} (x_{p+3})^{p+1} + \cdots + a_{i,p+3}^{(p,n)} = (x_{p+3})^{n+p+3-i} . \end{cases}$$

Then we conclude that

$$a_{i,j}^{(p,n)} = \frac{\det V_p^{(i,j)}}{\det V_p}$$

for each $i, j = 1, 2, \ldots, p+3$. $\square$

Thus by Theorem 2.2 and the matrix $(A_p)^n$, we have the following useful result for the Pell-Pell p-numbers.

Corollary 2.3. *Let p be a positive integer such that $p \geq 3$ and let $P_n^{P,p}$ be the nth element of Pell-Pell p-number, then*

$$P_n^{P,p} = \frac{\det V_p\,(p+3, 1)}{\det V_p}$$

and

$$P_n^{P,p} = -\frac{\det V_p\,(p+2, p+3)}{\det V_p}$$

for $n \geq 1$.

It is easy to see that the generating function of the Pell-Pell p-sequence $\left\{P_n^{P,p}\right\}$ is as follows:

$$g^{(p)}(x) = \frac{x^{p+2}}{1 - 4x + 3x^2 + 2x^3 - x^{p+1} + 2x^{p+2} + x^{p+3}},$$

where p$\geqslant 3$.

Then we can give an exponential representation for the Pell-Pell p-numbers by the aid of the generating function with the following Theorem.

Theorem 2.4. *The Pell-Pell p-sequences $\left\{P_n^{P,p}\right\}$ have the following exponential representation:*

$$g^{(p)}(x) = x^{p+2}\exp\left(\sum_{i=1}^{\infty}\frac{(x)^i}{i}\left(4-3x-2x^2+x^p-2x^{p+1}-x^{p+2}\right)^i\right),$$

where p$\geqslant 3$.

Proof. Since

$$\ln g^{(p)}(x) = \ln x^{p+2} - \ln\left(1-4x+3x^2+2x^3-x^{p+1}+2x^{p+2}+x^{p+3}\right)$$

and

$$-\ln\left(1-4x+3x^2+2x^3-x^{p+1}+2x^{p+2}+x^{p+3}\right) = -[-x\left(4-3x-2x^2+x^p-2x^{p+1}-x^{p+2}\right) - \frac{1}{2}x^2\left(4-3x-2x^2+x^p-2x^{p+1}-x^{p+2}\right)^2 - \cdots - \frac{1}{i}x^i\left(4-3x-2x^2+x^p-2x^{p+1}-x^{p+2}\right)^i - \cdots]$$

it is clear that

$$g^{(p)}(x) = x^{p+2}\exp\left(\sum_{i=1}^{\infty}\frac{(x)^i}{i}\left(4-3x-2x^2+x^p-2x^{p+1}-x^{p+2}\right)^i\right)$$

by a simple calculation, we obtain the conclusion. □

Let $K(k_1,k_2,\ldots,k_v)$ be a $v\times v$ companion matrix as follows:

$$K(k_1,k_2,\ldots,k_v) = \begin{bmatrix} k_1 & k_2 & \cdots & k_v \\ 1 & 0 & & 0 \\ \vdots & \ddots & & \vdots \\ 0 & \cdots & 1 & 0 \end{bmatrix}.$$

Theorem 2.5. *(Chen and Louck [4]) The (i,j) entry $k_{i,j}^{(n)}(k_1,k_2,\ldots,k_v)$ in the matrix $K^n(k_1,k_2,\ldots,k_v)$ is given by the following formula:*

$$k_{i,j}^{(n)}(k_1,k_2,\ldots,k_v) = \sum_{(t_1,t_2,\ldots,t_v)}\frac{t_j+t_{j+1}+\cdots+t_v}{t_1+t_2+\cdots+t_v}\times\binom{t_1+\cdots+t_v}{t_1,\ldots,t_v}k_1^{t_1}\cdots k_v^{t_v} \tag{2}$$

where the summation is over nonnegative integers satisfying $t_1+2t_2+\cdots+vt_v = n-i+j$, $\binom{t_1+\cdots+t_v}{t_1,\ldots,t_v} = \frac{(t_1+\cdots+t_v)!}{t_1!\cdots t_v!}$ is a multinomial coefficient, and the coefficients in (2) are defined to be 1 if $n=i-j$.

Then we can give other combinatorial representations than for the Pell-Pell p-numbers by the following Corollary.

Corollary 2.6. *i.*

$$P_n^{P,p} = \sum_{(t_1,t_2,\ldots,t_{p+3})} \binom{t_1+t_2+\cdots+t_{p+3}}{t_1,t_2,\ldots,t_{p+3}} 4^{t_1}(-3)^{t_2}(-2)^{t_3+t_{p+2}}(-1)^{t_{p+3}},\ (n \geq 1)$$

where the summation is over nonnegative integers satisfying $t_1 + 2t_2 + \cdots + (p+3)\,t_{p+3} = n - p - 2$.

ii.

$$P_n^{P,p} = -\sum_{(t_1,t_2,\ldots,t_{p+3})} \frac{t_{p+3}}{t_1+t_2+\cdots+t_{p+3}} \times \binom{t_1+t_2+\cdots+t_{p+3}}{t_1,t_2,\ldots,t_{p+3}} 4^{t_1}(-3)^{t_2}(-2)^{t_3+t_{p+2}}(-1)^{t_{p+3}}$$

$n \geq 1$, *where the summation is over nonnegative integers satisfying* $t_1 + 2t_2 + \cdots + (p+3)\,t_{p+3} = n + 1$.

Proof. If we take $i = p+3,\ j = 1$for the case i. and $i = p+2,\ j = p+3$ for the case ii. in Theorem 2.5, then we can directly see the conclusions from $(A_p)^n$. □

Now we consider the permanental representations for the Pell-Pell p-numbers.

Definition 2.1. *A $u \times v$ real matrix $M = [m_{i,j}]$ is called a contractible matrix in the k^{th} column (resp. row.) if the k^{th} column (resp. row.) contains exactly two non-zero entries.*

Suppose that $x_1,\ x_2,\ \ldots,x_u$ are row vectors of the matrix M. If M is contractible in the k^{th} column such that $m_{i,k} \neq 0, m_{j,k} \neq 0$ and $i \neq j$, then the $(u-1)\times(v-1)$ matrix $M_{ij:k}$ obtained from M by replacing the i^{th} row with $m_{i,k}x_j + m_{j,k}x_i$ and deleting the j^{th} row. The k^{th} column is called the contraction in the k^{th} column relative to the i^{th} row and the j^{th} row.

In [3], Brualdi and Gibson obtained that *per* $(M) =$ *per* (N) if M is a real matrix of order $\alpha > 1$ and N is a contraction of M.

Now we concentrate on finding relationships among the Pell-Pell p-numbers and the permanents of certain matrices which are obtained by using the generating matrix of the Pell-Pell p-numbers. Let $P_{m,p}^{P} = \left[p_{i,j}^{(p)}\right]$ be the $m \times m$

super-diagonal matrix, defined by

$$p_{i,j}^{(p)} = \begin{cases} 4 & \text{if } i=r \text{ and } j=r \text{ for } 1\le r\le m, \\ & \text{if } i=r+1 \text{ and } j=r \text{ for } 1\le r\le m-1 \\ 1 & \text{and} \\ & i=r \text{ and } j=r+p \text{ for } 1\le r\le m-p, \\ -1 & \text{if } i=r \text{ and } j=r+p+2 \text{ for } 1\le r\le m-p-2, \\ & \text{if } i=r \text{ and } j=r+2 \text{ for } 1\le r\le m-2 \\ -2 & \text{and} \\ & i=r \text{ and } j=r+p+1 \text{ for } 1\le r\le m-p-1, \\ -3 & \text{if } i=r \text{ and } j=r+1 \text{ for } 1\le r\le m-1, \\ 0 & \text{otherwise.} \end{cases}, \text{ for } m\ge p+3.$$

Then we have the following Theorem.

Theorem 2.7. *For $m \geqslant p+3$,*

$$perP_{m,p}^{P} = P_{m+p+2}^{P,p}.$$

Proof. Let us consider the matrix $P_{m,p}^{P}$ and let the equation be hold for $m \geqslant p+3$. Then we show that the equation holds for $m+1$. If we expand the $perP_{m,p}^{P}$ by the Laplace expansion of permanent with respect to the first row, then we obtain

$$perP_{m+1,p}^{P} = 4perP_{m,p}^{P} - 3perP_{m-1,p}^{P} - 2perP_{m-2,p}^{P} + perP_{m-p,p}^{P} - 2perP_{m-p-1,p}^{P} - perP_{m-p-2,p}^{P}.$$

Since

$$perP_{m,p}^{P} = P_{m+p+2}^{P,p},$$

$$perP_{m-1,p}^{P} = P_{m+p+1}^{P,p},$$

$$perP_{m-2,p}^{P} = P_{m+p}^{P,p},$$

$$perP_{m-p,p}^{P} = P_{m+2}^{P,p},$$

$$perP_{m-p-1,p}^{P} = P_{m+1}^{P,p}$$

and

$$perP_{m-p-2,p}^{P} = P_{m}^{P,p},$$

it is clear that $perP_{m+1,p}^{P} = P_{m+p+3}^{P,p}$. So the proof is complete. □

Let $R^{P}_{m,p}=\left[r^{(p)}_{i,j}\right]$ be the $m\times m$ matrix, defined by

$$r^{(p)}_{i,j}=\begin{cases} 4 & \text{if } i=r \text{ and } j=r \text{ for } 1\le r\le m-p-1,\\ & \text{if } i=r \text{ and } j=r \text{ for } m-p\le r\le m,\\ 1 & i=r+1 \text{ and } j=r \text{ for } 1\le r\le m-p-2\\ & \text{and}\\ & i=r \text{ and } j=r+p \text{ for } 1\le r\le m-p,\\ -1 & \text{if } i=r \text{ and } j=r+p+2 \text{ for } 1\le r\le m-p-2,\\ & \text{if } i=r \text{ and } j=r+2 \text{ for } 1\le r\le m-p-1\\ -2 & \text{and}\\ & i=r \text{ and } j=r+p+1 \text{ for } 1\le r\le m-p-1,\\ -3 & \text{if } i=r \text{ and } j=r+1 \text{ for } 1\le r\le m-p-1,\\ 0 & \text{otherwise.}\end{cases} \quad \text{for } m\ge p+3.$$

Then we have the following Theorem.

Theorem 2.8. *For $m\geqslant p+3$,*

$$perR^{P}_{m,p}=P^{P,p}_{m+1}.$$

Proof. Let us consider the matrix $R^{P}_{m,p}$ and let the equation hold for $m\ge p+3$. Then we show that the equation holds for $m+1$. If we expand $perR^{P}_{m,p}$ by the Laplace expansion of permanent according to the first row, then we obtain

$$perR^{P}_{m+1,p}=4perR^{P}_{m,p}-3perR^{P}_{m-1,p}-2perR^{P}_{m-2,p}+perR^{P}_{m-p,p}-2perR^{P}_{m-p-1,p}-perR^{P}_{m-p-2,p}.$$

Also, since

$$perR^{P}_{m,p}=P^{P,p}_{m+1},$$

$$perR^{P}_{m-1,p}=P^{P,p}_{m},$$

$$perR^{P}_{m-2,p}=P^{P,p}_{m-1},$$

$$perR^{P}_{m-p,p}=P^{P,p}_{m-p+1},$$

$$perR^{P}_{m-p-1,p}=P^{P,p}_{m-p}$$

and

$$perR^{P}_{m-p-2,p}=P^{P,p}_{m-p-1},$$

it is clear that $perR^{P}_{m+1,p}=P^{P,p}_{m+2}$. So the proof is complete. $\square$

Assume that $S_{m,p}^{P}=\left[s_{i,j}^{(p)}\right]$ be the $m\times m$ matrix, defined by

$$S_{m,p}^{P}=\begin{array}{c} \qquad\quad (m-p-1)\,\text{th} \\ \qquad\quad \downarrow \\ \left[\begin{array}{cccccc} 1 & \cdots & 1 & 0 & \cdots & 0 \\ 1 & & & & & \\ 0 & & & & & \\ \vdots & & & R_{m-1,p}^{P} & & \\ 0 & & & & & \\ 0 & & & & & \end{array}\right]\end{array}, \text{ for } m>p+3,$$

then we have the following results:

Theorem 2.9. *For $m>p+3$,*

$$perS_{m,p}^{P}=\sum_{i=0}^{m}P_{i}^{P,p}.$$

Proof. If we expand the $perS_{m,p}^{P}$ with respect to the first row, we write

$$perS_{m,p}^{P}=perS_{m-1,p}^{P}+perR_{m-1,p}^{P}.$$

Thus, by the results and an inductive argument, the proof is easily seen. □

A matrix M is called convertible if there is an $n\times n$ $(1,-1)$-matrix K such that $perM=\det(M\circ K)$, where $M\circ K$ denotes the Hadamard product of M and K.

Now we give relationships among the Pell-Pell p-numbers and the determinants of certain matrices which are obtained by using the matrix $P_{m,p}^{P}$, $R_{m,p}^{P}$ and $S_{m,p}^{P}$. Let $m>p+3$ and let H be the $m\times m$ matrix, defined by

$$H=\left[\begin{array}{cccccc} 1 & 1 & 1 & \cdots & 1 & 1 \\ -1 & 1 & 1 & \cdots & 1 & 1 \\ 1 & -1 & 1 & \cdots & 1 & 1 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 1 & \cdots & 1 & -1 & 1 & 1 \\ 1 & \cdots & 1 & 1 & -1 & 1 \end{array}\right].$$

Corollary 2.10. *For $m>p+3$,*

$$\det\left(P_{m,p}^{P}\circ H\right)=P_{m+p+2}^{P,p},$$

$$\det\left(R_{m,p}^{P}\circ H\right)=P_{m+1}^{P,p},$$

and

$$\det\left(S_{m,p}^{P}\circ H\right)=\sum_{i=0}^{m}P_{i}^{P,p}.$$

Proof. Since $perP_{m,p}^{P}=\det\left(P_{m,p}^{P}\circ H\right)$, $perR_{m,p}^{P}=\det\left(R_{m,p}^{P}\circ H\right)$ and $perS_{m,p}^{P}$ $=\det\left(S_{m,p}^{P}\circ H\right)$ for $m>p+3$, by Theorem 2.7, Theorem 2.8 and Theorem 2.9, we have the conclusion. □

Now we consider the sums of the Pell-Pell p-numbers. Let

$$S_n=\sum_{i=0}^{n}P_{i}^{P,p}$$

for $n\geq 1$ and let Y_P^p and $(Y_P^p)^n$ be the $(p+4)\times(p+4)$ matrices such that

$$Y_P^p=\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 & 0\\ 1 & & & & & & \\ 0 & & & & & & \\ \vdots & & & A_p & & & \\ 0 & & & & & & \\ 0 & & & & & & \\ 0 & & & & & & \end{bmatrix}, \text{ for } p\geqslant 3.$$

If we use induction on n, then we obtain

$$(Y_P^p)^n=\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 & 0\\ S_{n+p+1} & & & & & & \\ S_{n+p} & & & & & & \\ \vdots & & & (A_p)^n & & & \\ S_{n+1} & & & & & & \\ S_n & & & & & & \\ S_{n-1} & & & & & & \end{bmatrix}, \text{ for } p\geq 3.$$

References

[1] Y. Akuzum, O. Deveci and A. G. Shannon, *On the Pell p-circulant Sequences*, Notes on Number Theory and Disc. Math., **23** (2017), no. 2, 91-103.

[2] B. Bradie, *Extension and refinements of some properties of sums involving Pell number*, Missouri J.Math. Sci., **22** (2010), no. 1, 37-43.

[3] R. A. Brualdi and P. M. Gibson, *Convex polyhedra of doubly stochastic matrices I: applications of permanent function*, J. Combin. Theory, Series A, **22** (1997), no. 2, 194-230.

[4] W. Y. C. Chen and J. D. Louck, *The combinatorial power of the companion matrix*, Linear Algebra Appl., **232** (1996), 261-278.

[5] R. Devaney, *The Mandelbrot set and the Farey tree, and the Fibonacci sequence*, Amer. Math. Monthly, **106** (1999), 289-302.

[6] O. Deveci, *The Pell-Circulant Sequences and Their Applications*, Maejo Int. J. Sci. Technol., **10** (2016), no. 03, 284-293.

[7] O. Deveci and E. Karaduman, *On the Padovan p-numbers,* Hacettepe J. Math. Stat*.,* **46** (2017), no. 4, 579-592.

[8] D. D. Frey and J. A. Sellers, *Jacobsthal numbers and alternating sign matrices*, J. Integer Seq., **3** (2000), Article 00.2.3.

[9] N. Gogin and A. A. Myllari, *The Fibonacci-Padovan sequence and MacWilliams transform matrices*, Programing and Computer Software, published in Programmirovanie, **33** (2007), no. 2, 74-79.

[10] A. F. Horadam, *Jacobsthal representations numbers*, Fibonacci Quart., **34** (1996), 40-54.

[11] A. F. Horadam, *Applications of modified Pell numbers to representations*, Ulam Quart., **3** (1994), no. 1, 34-53.

[12] B. Johnson, *Fibonacci identities by matrix methods and generalisation to related sequences*, http://maths.dur.ac.uk/~dma0rcj/PED/fib.pdf, March 25, 2003.

[13] D. Kalman, *Generalized Fibonacci numbers by matrix methods*, Fibonacci Quart., **20** (1982), no. 1, 73-76.

[14] E. Kilic, *The generalized Pell (p,i)-numbers and their Binet formulas, combinatorial representations, sums*, Chaos, Solitons Fractals, **40** (2009), no. 4, 2047-2063.

[15] E. Kilic, *The Binet fomula, sums and representations of generalized Fibonacci p-numbers*, Eur. J. Comb., **29** (2008), 701–711.

[16] E. Kilic and D.Tasci, *The generalized Binet formula, representation and sums of the generalized order-k Pell numbers*, Taiwanese J. Math., **10** (2006), no. 6, 1661-1670.

[17] E. G. Kocer, *The Binet formulas for the Pell and Pell-Lucas p-numbers*, Ars Comb., **85** (2007), 3-17.

[18] F. Koken and D. Bozkurt, *On the Jacobsthal numbers by matrix methods*, Int. J. Contemp. Math. Sciences. **3** (2008), no. 13, 605-614.

[19] P. Lancaster and M. Tismenetsky, *The theory of matrices: with applications*, Elsevier, 1985.

[20] R. Lidl and H. Niederreiter, *Introduction to finite fields and their applications*, Cambridge UP, 1986.

[21] W. L. McDaniel, *Triangular numbers in the Pell sequence*, Fibonacci Quart., **34** (1996), 105-107.

[22] A. G. Shannon, P. G. Anderson and A. F. Horadam, *Properties of cordonnier Perrin and Van der Lan numbers*, Internat. J. Math. Ed. Sci. Tech., **37** (2006), no. 7, 825-831.

[23] A. G. Shannon, A. F. Horadam and P. G. Anderson, *The auxiliary equation associated with plastic number*, Notes on Number Theory and Disc. Math., **12** (2006), no. 1, 1-12.

[24] A.P. Stakhov, *A generalization of the Fibonacci Q-matrix*, Rep. Natl. Acad. Sci. Ukraine, **9** (1999), 46-49.

[25] A. P. Stakhov, B. Rozin, *Theory of Binet formulas for Fibonacci and Lucas p-numbers*, Chaos, Solitions Fractals, **27** (2006), 1162-77.

[26] I. Stewart, *Tales of a neglected number*, Sci. Amer. **274** (1996), 102-103.

[27] D. Tasci and M. C. Firengiz, *Incomplete Fibonacci and Lucas p-numbers*, Math. Comput. Modell., **52** (2010), 1763-1770.

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